

Dirac type operators for spin manifolds associated to congruence subgroups of generalized modular groups

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Abstract

Fundamental solutions of Dirac type operators are introduced for a class of conformally flat manifolds. This class consists of manifolds obtained by factoring out the upper half-space of \mathbb{R}^n by arithmetic subgroups of generalized modular groups. Basic properties of these fundamental solutions are presented together with associated Eisenstein series.

1 Introduction

A natural generalization to \mathbb{R}^n of the classical Cauchy-Riemann operator has proved to be the euclidean Dirac operator. Associated to this operator is a Cauchy Integral Formula and other natural analogues of basic results from one variable complex analysis. See for instance [4] and elsewhere. Further the euclidean Dirac operator has been used in understanding boundary value problems and aspects of classical harmonic analysis in \mathbb{R}^n . See for instance [21, 41] and elsewhere. This analysis together with its applications is known as Clifford analysis.

On the other hand Dirac operators have proved to be extremely useful tools in understanding geometry over spin manifolds. See for instance [30] and elsewhere. Basic aspects of Clifford analysis over spin manifolds have been developed in [6, 7, 37]. Further in [25, 26, 27, 33, 35, 39] and elsewhere it is illustrated

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that the context of conformally flat manifolds provide a useful setting for developing Clifford analysis.

Conformally flat manifolds are those manifolds which possess an atlas whose transition functions are Möbius transformations. Under this viewpoint conformally flat manifolds can be regarded as higher dimensional generalizations of Riemann surfaces, as pointed out for example in [35, 39]. These types of manifolds have been studied extensively in a number of contexts. See for instance [29, 40]. Following the classical work of N. H. Kuiper [29], one can construct a whole family of examples of conformally flat manifolds by factoring out a subdomain $U \subseteq \mathbb{R}^n$ by a Kleinian group Γ acting totally discontinuously on U .

Simple examples of conformally flat manifolds include spheres, hyperbolas, real projective space, cylinders, tori, and the Hopf manifolds $S^1 \times S^{n-1}$. In [33, 25, 26] explicit Clifford analysis techniques have been developed for these manifolds.

In this paper we treat some special examples of hyperbolic manifolds of higher genus with spinor structure. The manifolds that we consider arise from factoring out upper half-space in \mathbb{R}^n by an arithmetic congruence group, H , of the generalized modular group Γ_p . Γ_p is the group that is generated by p translation matrices ($p < n$) and the inversion matrix.

The associated manifolds are higher dimensional analogues of those classical Riemann surfaces that arise from factoring out the complex upper half-plane by the principal congruence subgroups of $SL(2, \mathbb{Z})$. In two real variables these are k -handled spheres.

Spinor sections on these manifolds can be constructed from automorphic forms on H . Using these automorphic forms we set up Cauchy kernels or fundamental solutions of the Dirac operators associated to these spin manifolds.

The basic theory of monogenic, Euclidean harmonic and more generally of k -genetic automorphic forms on this family of arithmetic groups is described in [23] using generalized Eisenstein series. However, as shown in [23], even in the monogenic case, the absolute convergence abscissa with respect to the generalized modular group of these generalized Eisenstein series is only $p < n - 2$. Here we overcome this convergence problem for the cases $p = n - 2$ and $p = n - 1$ by adapting a classical trick of Hecke, mentioned for example in [18].

Dirac operators associated to upper half space endowed with the hyperbolic metric and scalar perturbations of the hyperbolic metric and their associated hyperbolic Laplace operators have received steady attention. See for instance [1, 2, 13, 14, 15, 22, 31, 32, 38]. In this paper we introduce generalized Eisenstein series, with respect to the groups considered here, that are solutions to the these hyperbolic differential operators. We then use these series to introduce fundamental solutions to a particular hyperbolic Dirac operator and also for the hyperbolic Laplacian with respect to the manifolds considered here.

The lay out of the paper is as follows.

In Section 2 we introduce the background on Clifford algebras and Clifford analysis that we shall need here.

In Section 3 some of the geometry of these manifolds is described. In particular direct analogues of fundamental domains described for arithmetic groups in 2 and 3 real variables are introduced. The Ahlfors-Vahlen approach to describing Möbius transformations in n real variables is used to introduce isometric spheres to describe fundamental domains and their associated conformally flat manifolds.

In Section 4 for the cases $p = 1' \dots, n - 3$ generalized Eisenstein series are introduced over the universal covering space, $H^+(\mathbb{R}^n) = \{x \in \mathbb{R}^n : x_n > 0\}$, of these manifolds and the projection operator from $H^+(\mathbb{R}^n)$ to the manifold is used to produce non-trivial solutions to Dirac type equations over a spinor bundle. In this section the Hecke trick is adapted to introduce similar sections for the cases $p = n - 2$ and $p = n - 1$.

In Section 5 we construct special variants of Poincaré type series which induce the explicit Cauchy kernels for monogenic sections. Other generalized Poincaré series are used to explicitly determine fundamental solutions to higher order Dirac type operators on the manifolds. Basic properties of these fundamental solutions are investigated. In particular results mentioned in Section 3 are used to obtain Hardy space decompositions of the L^q spaces of compact strongly Lipschitz hypersurfaces lying in the manifolds considered here. This is for $q \in (1, \infty)$. Further the techniques used here are adapted to introduce operators of Calderon-Zygmund type in this context.

In Section 6 we develop the analogous results for k -hypergenic Eisenstein series and k -hyperbolic harmonic Eisenstein series. This function class includes hypermonogenic Eisenstein series when $k = n - 2$ and hyperbolic harmonic Eisenstein series also when $k = n - 2$. Analogous sections are set up over the corresponding conformally flat manifolds. In the second part of Section 6 fundamental solutions to the Dirac operator and hyperbolic Laplacian associated to these Eisenstein series are introduced and some of their basic properties are investigated. In particular we also provide a hypermonogenic Hardy space decomposition of the L^q space of a strongly Lipschitz hypersurface of the manifolds considered here. Again $q \in (1, \infty)$.

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2 Preliminaries

2.1 Clifford algebras

In this subsection we will introduce the basic information on Clifford algebras that we need in this paper. We shall regard Euclidean space, \mathbb{R}^n , as being embedded in the real, 2^n -dimensional, associative Clifford algebra, Cl_n , satisfying the relation $x^2 = -\|x\|^2$ for each $x \in \mathbb{R}^n$. In terms of the standard orthonormal basis e_1, \dots, e_n of \mathbb{R}^n this relation becomes the anti-commutation

$e_i e_j + e_j e_i = -2\delta_{ij}$ and a basis for Cl_n is given by

$$1, e_1, \dots, e_n, \dots, e_{j_1} \dots e_{j_r}, \dots, e_1 \dots e_n$$

where $j_1 < \dots < j_r$ and $1 \leq r \leq n$. Each non-zero vector in \mathbb{R}^n has a multiplicative inverse, $x^{-1} = \frac{-x}{\|x\|^2}$. Up to a sign this is the Kelvin inverse of a non-zero vector. For $a = a_0 + \dots + a_{1\dots n} e_1 \dots e_n \in Cl_n$ the nom of a is defined to be $\|a\| = (a_0^2 + \dots + a_{1\dots n}^2)^{\frac{1}{2}}$. The reversion anti-automorphism is defined by $\sim: Cl_n \rightarrow Cl_n : \sim e_{j_1} \dots e_{j_r} = e_{j_r} \dots e_{j_1}$. For $a \in Cl_n$ we write \tilde{a} for $\sim a$.

Any element $a \in Cl_n$ may be uniquely decomposed in the form $a = b + ce_n$, where $b, c \in Cl_{n-1}$. Based on this decomposition one defines the projection mappings $P: Cl_n \rightarrow Cl_{n-1}$ and $Q: Cl_n \rightarrow Cl_{n-1}$ by $Pa = b$ and $Qa = c$. Further we define Q^*a to be $e_n(Qa)e_n$. Note that if we define \hat{a} to be $b - ce_n$ then

$$P(a) = \frac{1}{2}(a + \hat{a}) \quad (1)$$

and

$$Q(a) = \frac{1}{2}(a - \hat{a}). \quad (2)$$

2.2 PDE's related to the Dirac operator in \mathbb{R}^n and hyperbolic space

Monogenic and k -genic functions.

The Dirac operator, D , in \mathbb{R}^n is defined to be $\frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \dots + \frac{\partial}{\partial x_n} e_n$. Suppose that U is a domain in \mathbb{R}^n and f and g are pointwise differentiable functions defined on U and taking values in Cl_n . The function f is called left monogenic or left Clifford holomorphic if it satisfies the equation $Df = 0$ on U . Similarly g is called right monogenic, or right Clifford holomorphic if it satisfies the equation $gD = 0$ on U . Here $gD := \sum_{j=1}^n \frac{\partial g(x)}{\partial x_j} e_j$. Due to the non-commutativity of Cl_n for $n > 1$, both classes of functions do not coincide with each other. However, f is left monogenic if and only if \tilde{f} is right monogenic. The left and right fundamental solution to the D -operator is called the Euclidean Cauchy kernel and has the form $G_1(x - y) = \frac{1}{\omega_n} \frac{x - y}{\|x - y\|^n}$. Here ω_n is the surface area of the unit sphere in \mathbb{R}^n . The Dirac operator factorizes the Euclidean Laplacian $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$, viz $D^2 = -\Delta$. Every real component of a monogenic function is hence harmonic. More generally, functions satisfying $D^k f = 0$ for a positive integer k are called left k -genic, while functions satisfying $gD^k = 0$ are called right k -genic. Note that when k is even, say $k = 2l$ for some positive integer l , then $D^{2l} = (-\Delta)^l$ and in this case left and right k -genic functions coincide.

The fundamental solution to the operator D^k for $k < n$ is $G_k(x - y) = C_k \frac{x - y}{\|x - y\|^{n-k+1}}$ when k is odd and $G_k(x - y) = \frac{C_k}{\|x - y\|^{n-k}}$ when k is even. Further C_k is a real positive constant chosen so that $D^{k-1} G_k(x - y) = G_1(x - y)$. In what follows we shall write $G_1(x - y)$ simply as $G(x - y)$.

Ahlfors-Vahlen matrices and iterated Dirac operators.

Following for example [3, 9], Möbius transformations in \mathbb{R}^n can be represented as

$$m : \mathbb{R}^n \cup \{\infty\} \rightarrow \mathbb{R}^n \cup \{\infty\} : m(x) = (ax + b)(cx + d)^{-1}$$

with coefficients a, b, c, d from Cl_n that can all be written as products of vectors from \mathbb{R}^n . Further $a\tilde{d} - b\tilde{c} \in \mathbb{R} \setminus \{0\}$ and $\tilde{a}c, \tilde{c}d, \tilde{d}b, \tilde{b}a \in \mathbb{R}^n$. These conditions are often called Vahlen conditions.

The set that consists of Clifford valued matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ whose coefficients satisfy the previously mentioned conditions is a group under matrix multiplication. It is called the general Ahlfors-Vahlen group. It is denoted by $GAV(\mathbb{R}^n)$.

The general linear Ahlfors-Vahlen group $GAV(\mathbb{R}^n)$ is a generalization of the general linear group $GL(2, \mathbb{R})$. The particular subgroup

$$SAV(\mathbb{R}^n) = \{M \in GAV(\mathbb{R}^n) \mid a\tilde{d} - b\tilde{c} = 1\}$$

is called the special Ahlfors-Vahlen group and it is purely generated by the inversion matrix and translation type matrices, as proved for instance in [9].

The projective special Ahlfors-Vahlen group is the group

$$PSAV(\mathbb{R}^n) \cong SAV(\mathbb{R}^n)/\{\pm I\}$$

where I is the identity matrix.

The subgroup $SAV(\mathbb{R}^{n-1})$ of $SAV(\mathbb{R}^n)$ has the special property that it acts transitively on the upper half space $H^+(\mathbb{R}^n) = \{x = x_1e_1 + \dots + x_n e_n \in \mathbb{R}^n : x_n > 0\}$.

Now assume that $m(x) = M < x >$, is a Möbius transformation represented in the above mentioned form. It is shown in [23] and elsewhere that if f is a left k -genic function in the variable $y = m(x) = (ax + b)(cx + d)^{-1}$, then the function $J_k(M, x)f(M < x >)$ is again left k -genic but now in the variable x .

Here

$$J_k(M, x) = \begin{cases} \frac{\widetilde{cx+d}}{\|cx+d\|^{n-k+1}} & k \text{ odd} \\ \frac{1}{\|cx+d\|^{n-k}} & k \text{ even} \end{cases}$$

In what follows we always restrict attention to the cases where $k < n$. This type of invariance for k -genic functions under Möbius transformations is seen as an automorphic invariance.

It may now be seen that if f is a function that is left k -genic on $H^+(\mathbb{R}^n)$ then so is $J_k(M, x)f(M < x >)$, for any $M \in SAV(\mathbb{R}^{n-1})$. In what follows we shall write $J(M, x)$ for $J_1(M, x)$.

k -hypergenic functions.

Left k -hypergenic functions are defined as the null-solutions to the system

$$M_k f := Df + \frac{kQ^* f}{x_n} = 0, \quad x_n \neq 0,$$

where $k \in \mathbb{R}$. This is a Hodge-Dirac equation for upper half space equipped with the metric $x_n^{\frac{-2k}{n-2}} \sum_{j=1}^n dx_j^2$. In the case $k = 0$ these are precisely left monogenic functions. A similar definition can be given for right k -hypergenic functions.

It is pointed out in [38] and elsewhere that

$$-M_k^2 f = (\Delta P f - \frac{k}{x_n} \frac{\partial P f}{\partial x_n}) + (\Delta Q f - \frac{k}{x_n} \frac{\partial Q f}{\partial x_n} + \frac{k}{x_n^2} Q f).$$

The operator $\Delta - \frac{k}{x_n} \frac{\partial}{\partial x_n}$ is the Laplacian for $H^+(\mathbb{R}^n)$ with respect to the metric $x_n^{\frac{-2k}{n-2}} \sum_{j=1}^n dx_j^2$. We will denote it by Δ_k and we shall call it the k -hyperbolic Laplacian. We will denote the operator $\Delta_k + \frac{k}{x_n^2}$ by W_k . Functions that are annihilated by the k -hyperbolic Laplacian will be called k -hyperbolic harmonic functions. When $k = n - 2$ this operator is the hyperbolic Laplacian and functions that are annihilated by this operator are called hyperbolic harmonic functions. The operators Δ_k and W_k are special cases of the Weinstein equation described in [1, 31] and elsewhere.

Under a Möbius transformation $y = M < x > = (ax + b)(cx + d)^{-1}$ a k -hypergenic function $f(y)$ is transformed to the k -hypergenic function

$$F(x) := K_k(M, x) f(M < x >), \quad (3)$$

where $K_k(M, x) = \frac{\widetilde{cx+d}}{\|cx+d\|^{n-k}}$. See for instance [15, 38]. The particular solutions associated to the case $k = n - 2$ coincide with the null-solutions to the hyperbolic Hodge-Dirac operator with respect to the hyperbolic metric on upper half space. These are often called hyperbolic monogenic functions or simply hypermonogenic functions [14, 32]. Note that when k is even then $K_k(M, x) = J_{k+1}(M, x)$. As explained in [14, 38] the basic hypermonogenic kernels for $H^+(\mathbb{R}^n)$ are given by

$$p(x, y) = \frac{1}{\omega_n} x_n^{n-2} y_n^{n-1} \frac{(x - y)}{\|x - y\|^n} e_n \frac{(x - \hat{y})}{\|x - \hat{y}\|^n}$$

and

$$q(x, y) = y_n^{n-2} \left(\frac{1}{\|x - \hat{y}\|^{n-2}} G(x - y) + \frac{1}{\|x - y\|^{n-2}} G(x - \hat{y}) \right) = y_n^{n-2} D_x H(x, y)$$

where D_x is the Dirac operator with respect to the variable x and $H(x, y) = \frac{1}{(n-2)\omega_n} \frac{1}{\|x - y\|^{n-2} \|x - \hat{y}\|^{n-2}}$.

These kernels provide analogues of the Cauchy kernel $G(x - y)$ for hypermonogenic functions. More precisely if $f(x)$ is a left hypermonogenic function defined on a domain $U \subset \mathcal{F}_p[N]$ then

$$f(y) = P \left(\int_{\partial V} p(x, y) \frac{n(x)}{x_n^{n-2}} f(x) d\sigma(x) \right) + Q \left(\int_{\partial V} q(x, y) n(x) f(x) d\sigma(x) \right) \quad (4)$$

where V is a subdomain of U whose closure is compact and lies in U . Further $y \in V$.

Using Equations (1) and (2) it is shown in [14] that Equation (4) can be rewritten as

$$f(y) = y_n^{n-1} \left(\int_{\partial V} E(x, y) n(x) f(x) d\sigma(x) + \int_{\partial V} F(x, y) \hat{n}(x) \hat{f}(x) d\sigma(x) \right)$$

where $E(x, y) = \frac{2^{n-1}}{\|x-y\|^{n-2}} G(x-y)$ and $F(x, y) = \frac{2^{n-1}}{\|x-y\|^{n-2}} G(\hat{x}-y)$.

It is shown in [1] that if $M \in SAV(\mathbb{R}^{n-1})$ then a k hyperbolic harmonic function $f(y)$ is transformed to the k -hyperbolic harmonic function $L_k(M, x)f(M < x >)$ where $y = M < x >$ and $L_k(M, x) = \frac{1}{\|cx+d\|^{n-k-2}}$.

Following [13] and [31] respectively we also have by direct computation:

Proposition 1 (i) A function f is left k -hypergenic if and only if $\frac{f}{x_n^k} e_n$ is left $-k$ -hypergenic.

(ii) A function h is k -hyperbolic harmonic if and only if $x_n^{-k} h(x)$ is a solution to the Weinstein equation $W_{-k}u = 0$.

It should also be noted that x_n^{k+1} is a solution to the equation $\Delta'_k u = 0$. This result may be obtained by direct computation.

In [1] it is shown that if $u(y)$ is a solution to $\Delta'_k u = 0$ then so is $L_k(M, x)u(M < x >)$ where $y = M < x >$.

A function that is annihilated by the operator Δ'_k is an eigenfunction of the operator $x_n^2 \Delta_k$ with eigenvalue $-k$.

3 Subgroups of generalized modular groups, their fundamental domains and associated conformally flat manifolds

3.1 Arithmetic subgroups of the Ahlfors-Vahlen group

Arithmetic subgroups of the special Ahlfors-Vahlen group that act totally discontinuously on $H^+(\mathbb{R}^n)$ are for instance considered in [9, 11, 12] and for the three dimensional case in [34].

First let us introduce the ring

$$\mathcal{O}_p := \sum_{A \subseteq P(1, \dots, p)} \mathbb{Z} e_A \quad p \leq n-1.$$

This ring of course lies in the subalgebra Cl_p . In what follows, let

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T_{e_1} := \begin{pmatrix} 1 & e_1 \\ 0 & 1 \end{pmatrix}, \dots, T_{e_p} := \begin{pmatrix} 1 & e_p \\ 0 & 1 \end{pmatrix}.$$

We recall, cf. [23]:

Definition 1 For $p < n$, the special hypercomplex modular groups are defined to be $\Gamma_p := \langle J, T_{e_1}, \dots, T_{e_p} \rangle$. For a positive integer N the associated principal congruence subgroups of Γ_p of level N are then given by

$$\Gamma_p[N] := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_p \mid a-1, b, c, d-1 \in N\mathcal{O}_p \right\}.$$

As the group $SL(2, \mathbb{Z})$ is generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ it can be seen that Γ_p is a natural generalization to \mathbb{R}^n of $SL(2, \mathbb{Z})$. It follows that the group $\Gamma_p[N]$ is a natural generalization of arithmetic subgroups of $SL(2, \mathbb{Z})$ described in [18] and elsewhere.

One can readily adapt arguments given in [18] to see that the group Γ_p acts totally discontinuously on $H^+(\mathbb{R}^n)$. As $\Gamma_p[N] \subset \Gamma_p$ for each positive integer N it follows that $\Gamma_p[N]$ also acts totally discontinuously on $H^+(\mathbb{R}^n)$. Consequently $H^+(\mathbb{R}^n)/\Gamma_p[N]$ is a conformally flat manifold. We will denote this manifold by $\mathcal{M}_p[N]$.

3.2 Fundamental domains and their associated conformally flat manifolds

It is known that all discrete arithmetic subgroups Γ of Ahlfors-Vahlen's group $SAV(\mathbb{R}^{n-1})$ possess a fundamental domain in $H^+(\mathbb{R}^n)$. See for example [10] where the general n -dimensional case is treated in condensed form. A very detailed description of the fundamental domains in the particular three-dimensional case can be found in [12]. Just to recall: For a discrete subgroup, Γ , of $SAV(\mathbb{R}^{n-1})$ a fundamental domain $\mathcal{F}(\Gamma) \subset H^+(\mathbb{R}^n)$ is a relatively closed domain in $H^+(\mathbb{R}^n)$ with the two properties:

1. $H^+(\mathbb{R}^n) = \cup_{M \in \Gamma} M\langle \mathcal{F}(\Gamma) \rangle$
2. $\text{int } \mathcal{F}(\Gamma) \cap M\langle \text{int } \mathcal{F}(\Gamma) \rangle \neq \emptyset, M \in \Gamma, \implies M = \pm I$.

One can describe the geometry of a fundamental domain of a discrete group Γ acting on the upper half space in \mathbb{C} in terms of the set of its isometric circles [5, 17].

The isometric circle for a Möbius transformation $\frac{az+b}{cz+d}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ with $c \neq 0$ would be the circle

$$\{z \in H^+(\mathbb{C}) : \|cz + d\| = 1\}.$$

In [5] it is shown that the role played by isometric circles in introducing fundamental domains for the arithmetic subgroups of the modular group $SL(2, \mathbb{Z})$, also referred to as Ford domains, can be carried over to upper half space in \mathbb{R}^n . In this context the isometric circles are replaced by isometric spheres. So given a Möbius transformation $(ax + b)(cx + d)^{-1}$ induced by a matrix

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SAV(\mathbb{R}^n)$ with $c \neq 0$ we have that

$$\left\| \frac{\partial[(ax+b)(cx+d)^{-1}]}{\partial x_j} \right\| = \frac{1}{\|cx+d\|^2} \quad \text{for } j = 1, \dots, n.$$

Consequently for such a Möbius transformation its isometric sphere is defined to be the sphere $\{x \in \mathbb{R}^n : \|cx+d\| = 1\}$. This is a sphere in \mathbb{R}^n centered at $-dc^{-1}$ and of radius $\|dc^{-1}\|$. Let us denote this isometric sphere by S_M . Following [5, 42] in complete parallel to the complex case it may be shown that $M < S_M > = S_{M^{-1}}$. In analogy to the complex case we say that the Möbius transformation $(ax+b)(cx+d)^{-1}$ induced by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SAV(\mathbb{R}^n)$ with $c \neq 0$ is hyperbolic if the isometric spheres S_M and $S_{M^{-1}}$ are external. It is elliptic if they intersect and $M < x >$ is parabolic if S_M and $S_{M^{-1}}$ are tangent. It should be noted that as in the case of isometric circles the interior $B(-dc^{-1}, \|dc^{-1}\|)$, of the isometric sphere S_M is mapped via M to the exterior of the isometric sphere $S_{M^{-1}}$.

Suppose H is a discrete subgroup of $SAV(\mathbb{R}^{n-1})$ that acts totally discontinuously on \mathbb{R}^n , and $\mathcal{F}(H)$ is its associated fundamental domain. If one glues exactly the equivalence points of the boundary parts $\partial\mathcal{F}_p$ under the group action together, then one obtains a conformally flat manifold. This results from the factorization $H^+(\mathbb{R}^n)/H$. These manifolds belong to the general class of hyperbolic manifolds.

Let us give some simple examples.

For illustration, let us consider as a concrete example the group $\Gamma_1[4]$. This group is isomorphic to the classical arithmetic group $\Gamma[4]$. In [5, 16] it is shown that one fundamental domain for $H^+(\mathbb{C})/\Gamma[4]$ is the open set in $H^+(\mathbb{C})$ bounded between the lines $x = -1$ and $x = 3$ and the 8 isometric circles of radius $\frac{1}{4}$ and centered on the x axis at $\frac{-3}{4}, \dots, 2\frac{3}{4}$ respectively. These are the isometric circles for the Möbius transformations

$$M_1 < z > = \frac{5z+4}{-4z-3}, \dots, M_8 < z > = \frac{-3z+8}{4z-11}, \quad M_1, \dots, M_8 \in \Gamma[4],$$

respectively. To obtain the corresponding Riemann surface one identifies the two lines $\{z = x+iy : x = -1\}$ and $\{z = x+iy : x = 3\}$ and the isometric semicircle lying in upper half space associated with $M_1 < z >$ with the isometric semicircle associated with $M_8 < z >$. Further for $j = 2, 4, 6$ the isometric semicircles lying in upper half plane and associated with the Möbius transformation $M_j < z >$ is identified with the isometric semicircle lying in upper half space and associated with $M_{j+1} < z >$. If we consider the action of $\Gamma_1[4]$ on $H^+(\mathbb{R}^n)$, then the isometric semicircles associated with M_1, \dots, M_8 are replaced by the isometric hemispheres C_1, \dots, C_8 lying in upper half space $H^+(\mathbb{R}^n)$ and now associated with the Möbius transformations

$$M'_1 < x > = (5x+4e_1)(4e_1x-3)^{-1}, \dots, M'_8 < x > = (-3x+8e_1)(-4e_1x-11)^{-1}.$$

Furthermore the lines $x = -1$ and $x = 3$ are replaced by the half hyperplanes $C_9 = \{x \in H^+(\mathbb{R}^n) : x_1 = -1\}$ and $C_{10} = \{x \in H^+(\mathbb{R}^n) : x_1 = 3\}$. A fundamental domain associated to the action of $\Gamma_1[4]$ on $H^+(\mathbb{R}^n)$ is the domain in $H^+(\mathbb{R}^n)$ with boundary $C_1 \cup \dots \cup C_{10}$. This domain is unbounded in the variables x_2, \dots, x_{n-1} and x_n . Consequently the boundary of this fundamental domain has nontrivial intersection with unbounded open subsets of $\partial H^+(\mathbb{R}^n)$.

To obtain the conformally flat manifold $H^+(\mathbb{R}^n)/\Gamma_1[4]$ we now identify C_9 with C_{10} , C_1 with C_8 and C_j with C_{j+1} for $j = 2, 4, 6$.

The group Γ_2 is isomorphic to the special linear group over the Gaussian integers $SL_2(\mathbb{Z}[e_1])$, where $\mathbb{Z}[e_1]$ is the lattice of Gaussian integers $\{p + qe_1 : p, q \in \mathbb{Z}\}$. This group acts totally discontinuously on the modified upper half-space

$$H^+(\mathbb{R} \oplus \mathbb{R}^{n-1}) = \{x_0 + x_1 e_1 + \dots + x_{n-1} e_{n-1} : x_{n-1} > 0\}.$$

Of course the group $SL_2(\mathbb{Z}[e_1])$ is essentially the Picard group $SL_2(\mathbb{Z}[i])$, the subgroup of $SL_2(\mathbb{C})$ with coefficients in the lattice of Gaussian integers $\{p + qi : p, q \in \mathbb{Z}\}$. Following for example [28], one may determine that one fundamental domain for $SL_2(\mathbb{Z}[e_1])$ is $F_2 := \{x = x_0 + x_1 e_1 + \dots + x_{n-1} e_{n-1} \in H^+(\mathbb{R} \oplus \mathbb{R}^{n-1}) \mid \|x\| \geq 1, 0 < x_j < \frac{1}{2}, j = 0, 1\}$.

It follows from our constructions that standard methods of constructing fundamental domains for $\Gamma_1[N]$ in one complex variable and for $\Gamma_2[N]$ in three real variables described in [17, 16, 28] and elsewhere extend readily to give us examples of fundamental domains for $\Gamma_1[N]$ and $\Gamma_2[N]$ in n -real variables.

Notice that for a given discrete subgroup $\Gamma_p[N]$ there are infinitely many choices of fundamental domain. If U is a particular fundamental domain for $\Gamma_p[N]$ then so is $M < U >$ for each $M \in \Gamma_p[N]$. The fundamental domain we will choose to work with will be denoted by $\mathcal{F}_p[N]$ and it will be a fundamental domain for $\Gamma_p[N]$ that lies in the set $U(p, N) = \{x \in H^+(\mathbb{R}^n) : -\frac{N}{2} \leq x_i \leq \frac{N}{2} \text{ for } i = 1, \dots, p\}$, and it will be unbounded in the variables x_{p+1}, \dots, x_n .

4 Monogenic Eisenstein series on $\Gamma_p[N]$ and the Hecke trick

$H^+(\mathbb{R}^n)$ is the universal covering space of the class of manifolds $\mathcal{M}_p[N]$ we are considering here. As a consequence, there exists a well-defined projection map $p : H^+(\mathbb{R}^n) \rightarrow \mathcal{M}_p[N] : x \mapsto x \pmod{\Gamma_p[N]}$. Let us write x' for $p(x)$ where $x \in H^+(\mathbb{R}^n)$. For each open set $U \subseteq H^+(\mathbb{R}^n)$ we write U' for $p(U)$ which in turn is an open subset of $\mathcal{M}_p[N]$. Further for each set $S \subset \mathcal{F}_p[N]$ we will write S' for $p(S)$.

In this section we give some elementary examples of automorphic forms on the groups $\Gamma_p[N]$ (with $N \geq 3$). More precisely, the examples we give here are invariant under the action of $\Gamma_p[N]$ up to the particular automorphic weight factor $J(M, x)$. These then project down to form non-trivial examples of monogenic sections on the associated families of conformally flat manifolds. These sections take values in a fixed spinor bundle F_1 .

More precisely this bundle is constructed over $\mathcal{M}_p[N]$ by identifying each pair (x, X) with $(M \langle x \rangle, J(M, x)X)$ for every $M \in \Gamma_p[N]$ where $x \in H^+(\mathbb{R}^n)$, and $X \in Cl_n$.

It should be noted that as $(ax + b)(cx + d)^{-1} = (-ax - b)(-cx - d)^{-1}$ then usually there is an ambiguity of sign in the previous identification of X with $J(M, x)X$. However as $-I$ does not belong to the group $\Gamma_p[N]$ for $N > 2$ there is no ambiguity of sign in this particular case and so the bundle F_1 is globally well defined.

Definition 2 Suppose that U' is a domain in $\mathcal{M}_p[N]$ and $f : U' \rightarrow F_1$ is a section that locally is annihilated on the left by the Dirac operator, D , then f is called a (left) monogenic section.

In the previous definition we locally used the Dirac operator, D . In fact we have introduced a Dirac operator $D_{\mathcal{M}_p[N]}$ that acts globally on sections in F_1 . This Dirac operator is in fact the Atiyah-Singer operator, or Atiyah-Singer-Dirac operator, cf. [30].

In [23] it is noted that when $N = 1$ or $N = 2$ the matrix $-I$ belongs to Γ_N . Consequently any function that satisfies $f(x) = J(M, x)f(M \langle x \rangle)$ for all $M \in \Gamma_p[N]$ and each $x \in H^+(\mathbb{R}^n)$ must satisfy $f(x) = -f(x)$ and so vanishes identically. For this reason we shall unless otherwise specified work in this paper with the cases $N \geq 3$.

For $N \geq 3$, as $-I$ is not in $\Gamma_p[N]$ it is possible to construct non-trivial monogenic functions that satisfy $f(x) = J(M, x)f(M \langle x \rangle)$ for all $M \in \Gamma_p[N]$. To introduce these types of Eisenstein series we shall need the following convergence lemma which is proved in [23].

Lemma 1 (Convergence lemma) For all $\alpha > p + 1$ and all $N \in \mathbb{N}$ the series

$$\sum_{M: \Gamma_p[N] \backslash \mathcal{T}_p[N]} \frac{1}{\|cx + d\|^\alpha}$$

converges uniformly on each compact subset of $H^+(\mathbb{R})$. Here $M : \Gamma_p[N] \backslash \mathcal{T}_p[N]$ denotes a sum over representatives of the left coset space $\Gamma_p[N] \backslash \mathcal{T}_p[N]$.

Now following [23] we introduce the following generalized Eisenstein series.

Definition 3 For $p < n - 2$ the monogenic Eisenstein series attached to $\Gamma_p[N]$ are then defined by

$$E_{p,N}(x) = \sum_{M: \Gamma_p[N] \backslash \mathcal{T}_p[N]} J(M, x). \quad (5)$$

Here the notation $M : \Gamma_p[N] \backslash \mathcal{T}_p[N]$ means that the matrices M run through a system of representatives of right cosets in $\Gamma_p[N]$ modulo the translation group $\mathcal{T}_p[N]$. The convergence lemma insures the convergence of the Series (5).

Here we shall assume that each representative M from the left coset space $\Gamma_p[N] \backslash \mathcal{T}_p[N]$ is chosen so that $M(\mathcal{F}_p[N])$ lies in the set $U(p, N)$.

Since the identity matrix is a representative from the coset space $\Gamma_p[N] \backslash \mathcal{T}_p[N]$ and $J(I, x) = 1$ and as $\lim_{x \rightarrow \infty} J(M, x) = 0$ for each $M \in \Gamma_p[N] \backslash \mathcal{T}_p[N]$ with $M \neq I$ then $\lim_{x_n \rightarrow +\infty} E_{p,N}(e_n x_n) = 1$. Consequently the Series (5) does not vanish identically when $N \geq 3$.

One can directly verify by a rearrangement argument, [23], that the Series (5) satisfies the transformation rule

$$E_{p,N}(x) = J(M, x) E_{p,N}(M < x >) \quad \forall M \in \Gamma_p[N].$$

Hence the generalized Eisenstein series (5) project down to non-trivial monogenic sections with values in the spinor bundle F_1 over the manifold $\mathcal{M}_p[N]$.

Remark. If we extend the sum in Expression (5) to the whole group $\Gamma_p[N]$, the series would diverge. This is due to the fact that the summation of the expressions $J(M, x)$ over the translation group $\mathcal{T}_p[N]$ diverges.

By adapting the so called Hecke trick we can get convergent Eisenstein series in the remaining two cases $p = n - 2$ and $p = n - 1$. This is a classical method to introduce Eisenstein and Poincaré series of lower weight. See for instance [18]. In analogy to the one complex variable and several complex variable cases let us introduce the following adapted Eisenstein series

$$E_{p,N}(x)(x, s) = \sum_{M: \mathcal{T}_p[N] \backslash \Gamma_p[N]} \left(\frac{x_n}{\|cx + d\|^2} \right)^s J(M, x), \quad (6)$$

where s is a complex auxiliary parameter. In view of the convergence lemma these series are a priori normally convergent and complex-analytic in s whenever the real part of s is greater than $-n + 2 + p$.

Whenever the real part of s is greater than $-n + 2 + p$, the expressions $\left(\frac{x_n}{\|cx + d\|^2} \right)^s J(M, x)$ are not monogenic in the vector variable x . However, one can show:

Proposition 2 *The series $E_{p,N}(x, s)$ possesses a continuous extension to the complete complex s -semi-plane $\{s = u + iv \in \mathbb{C} : v \geq 0\}$.*

The limit

$$E_{p,N}(x) := \lim_{s \rightarrow 0^+} \sum_{M: \mathcal{T}_p[N] \backslash \Gamma_p[N]} \left(\frac{x_n}{\|cx + d\|^2} \right)^s J(M, x) \quad (7)$$

will then provides us with a left monogenic Eisenstein series for the larger groups $\Gamma_{n-2}[N]$ and $\Gamma_{n-1}[N]$ which then of course has the desired invariance behavior under the respective actions of $\Gamma_{n-2}[N]$ and $\Gamma_{n-1}[N]$.

The proof of the continuous extension of the series $E_{p,N}(x, s)$ towards $s \rightarrow 0^+$ can be done in the same way as the classical proof of the holomorphic Hilbert

Eisenstein series in several complex variables as presented for instance in [18] pp. 165–172. Hence, we leave the detailed proof as an exercise for the reader and restrict ourselves to only present here the features of the proof which are different from the classical proof.

Without loss of generality we focus on the case $p = n - 1$, since the series $E_{n-2,N}(x, s)$ are subseries of $E_{n-1,N}(x, s)$. As in the classical case, one expands the $\Gamma_{n-1}[N]$ -periodic function $E_{n-1,N}(x, s)$ into a Fourier series. This can be done easily by first expanding the $n - 1$ -fold periodic subseries

$$\varepsilon_{n-1,N}(x, s) = \sum_{\underline{g} \in N\mathbb{Z}^{n-1}} G(x + \underline{g}) \left(\frac{x_n}{\|x + \underline{g}\|^2} \right)^s \quad (8)$$

into a Fourier series of the form

$$\sum_{\underline{g} \in \frac{1}{N}\mathbb{Z}^{n-1}} \alpha(x_n, \underline{g}; s) e^{2\pi i \langle \underline{g}, \underline{x} \rangle}$$

and showing that this series has a continuous extension as $s \rightarrow 0^+$. The Fourier coefficients are basically given (up to a constant) by the following integral (putting $\underline{x} := x_1 e_1 + \cdots x_{n-1} e_{n-1}$):

$$\alpha(x_n, \underline{g}; s) = \int_{\mathbb{R}^{n-1}} D_y \left[\frac{x_n^s}{\|x + y\|^{n-2+2s}} e^{-2\pi i \langle x+y, \underline{g} \rangle} \right] d\underline{x}. \quad (9)$$

We consider

$$\begin{aligned} &= \int_{\mathbb{R}^{n-1}} \frac{x_n^s}{(\|\underline{x}\|^2 + x_n^2)^{\frac{n-2}{2}+s}} e^{-2\pi i \langle \underline{x}, \underline{g} \rangle} d\underline{x} \\ &= \int_{S_{n-3}} \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} \frac{x_n^s r^{n-2}}{(r^2 + x_n^2)^{\frac{n-2}{2}+s}} e^{-2\pi i r \|\underline{g}\| \cos(\theta)} \sin^{n-3}(\theta) dr d\theta dS_{n-3} \\ &= \omega_{n-3} \int_{r=0}^{\infty} \frac{x_n^s r^{n-2}}{(r^2 + x_n^2)^{\frac{n-2}{2}+s}} r^{-\frac{n-3}{2}} J_{\frac{n-3}{2}}(2\pi r \|\underline{g}\|) dr \end{aligned} \quad (10)$$

$$= \frac{2^{\frac{3-2s}{2}} (2\pi \|\underline{g}\|)^{\frac{2s-1}{2}} \pi^{\frac{n-1}{2}} x_n^{\frac{1}{2}}}{\Gamma(\frac{n+2s}{2} - 1)} K_{\frac{1-2s}{2}}(2\pi \|\underline{g}\| x_n) \quad (11)$$

where we applied the substitution $r := \|\underline{x}\|$ and where θ is the angle between \underline{x} and \underline{g} . $J_{n-3/2}$ and $K_{1-2s/2}$ are the standard Bessel functions of first and second type.

The function $x_n^{\frac{1}{2}} K_{\frac{1-2s}{2}}$ is entire-complex analytic in s . In particular, it is continuous at $s = 0$. This establishes the existence of the limit $s \rightarrow 0^+$. Therefore,

we may in particular interchange the application of the Dirac operator in (9) with the integration process.

The rest of the analyticity proof can now be adapted directly from the classical proof presented in [18] pp. 165. One re-expresses the complete series $E_{n-1,N}(x)$ as a series over the Fourier series of $\varepsilon_{n-1,N}(x, s)$.

As the term $\lim_{s \rightarrow 0} \sum_{M_1, \dots, M_r} (\frac{x_n}{\|cx+d\|})^s J(M, x)$ is left monogenic, where summation is taken over any finite subset $\{M_1, \dots, M_r\}$ of $\Gamma_k[N] \setminus \mathcal{T}_k[N]$, it follows that the functions $E_{n-2,N}(x)$ and $E_{n-1,N}(x)$ are left monogenic.

The projections of the functions $E_{n-2,N}(x)$ and $E_{n-1,N}(x)$ to sections over $\mathcal{M}_p[N]$ thus provide us also with non-trivial examples of monogenic sections on the manifolds $\mathcal{M}_p[N]$ with $p = n - 2$ and $p = n - 1$.

We shall need the following, cf. [23]:

Definition 4 *Suppose for k odd that $N \geq 3$ and for k even $N \in \mathbb{N}$. Suppose also that $p < n$ then the k -genic Eisenstein series attached to $\Gamma_p[N]$ are defined to be the series*

$$E_{p,N,k}(x) = \sum_{M: \Gamma_p[N] \setminus \mathcal{T}_p[N]} J_k(M, x). \quad (12)$$

Convergence of the Series (12) follows from Lemma 2.

5 Cauchy kernels for monogenic and k -genic sections

Here we use the Eisenstein series introduced in Definition 3 to introduce an explicit formula for the Cauchy kernel or fundamental solution to the Dirac operator on the manifolds $\mathcal{M}_p[N]$ with $p < n - 2$. We then use the Hecke trick introduced in the previous section to introduce the fundamental solutions for the cases $p = n - 2$ and $p = n - 1$. We also introduce fundamental solutions to the analogues of the operators D^k for $k < n$ over $\mathcal{M}_p[N]$ using the Eisenstein series introduced in Definition 4. Further Calderon-Zygmund operators in this context are introduced.

We now proceed to prove the first main result of this section:

Proposition 3 *For $p < n - 2$ and for each $x \in \mathcal{F}_p[N]$ the series*

$$\sum_{T \in \mathcal{T}_p[N]} \sum_{M: \Gamma_p[N] \setminus \mathcal{T}_p[N]} J(TM, x) G(y - TM < x >). \quad (13)$$

converges uniformly on any compact subset K of $\mathcal{F}_p[N] \setminus \{x\}$.

Remark. Notice that the second series $\sum_{M: \Gamma_p[N] \setminus \mathcal{T}_p[N]} J(TM, x) G(y - TM < x >)$ depends on the particular choice of the system of representatives of right

cosets in $\Gamma_p[N]$ modulo $\mathcal{T}_p[N]$ since the function $G(y - TM < x >)$ is not invariant under $\mathcal{T}_p[N]$. This notation is understood in the sense that one has to specify first a particular system of representatives. The whole double sum however then gets again independent of that particular choice since the first sum extends over the whole translation group $\mathcal{T}_p[N]$.

Proof: It should first be noted that $J(TM, x) = J(M, x)$ for each $T \in \mathcal{T}_p[N]$ and each $M \in \Gamma_p[N]$. So from Lemma 1 for each fixed $T \in \mathcal{T}_p[N]$ the series $\sum_{M \in \Gamma_p[N] \setminus \mathcal{T}_p[N]} J(TM, x)$ converges uniformly on K for $p < n - 2$. Further the series (13) can be rewritten as

$$\sum_{T \in \mathcal{T}_p[N]} \sum_{M \in \Gamma_p[N] \setminus \mathcal{T}_p[N]} J(M, x) G(y - TM < x >).$$

In [23] it is shown that the series $\sum_{T \in \mathcal{T}_p[N]} G(y - T < x >)$ converges uniformly

for $p < n - 1$. As each representative M from $M \in \Gamma_p[N]$ is chosen so that $M(\mathcal{F}_p[N]) \subset U(p, N)$ then for each $T_m \in \mathcal{T}_p[N] \setminus \{I\}$ and each $y \in \mathcal{F}_p[N]$ we have that $\|G(y - T_m M < x >)\| \leq \|G(w - \underline{m})\|$ where $T_m < x > = x + \underline{m}$ and $\underline{m} \in \mathbb{Z}e_1 \dots \mathbb{Z}e_p$. Consequently it follows from Lemma 1 that the subseries

$$\sum_{T \in \mathcal{T}_p[N] \setminus \{I\}} \sum_{M \in \Gamma_p[N] \setminus \mathcal{T}_p[N]} J(TM, x) G(y - TM < x >)$$

is uniformly convergent on K for $p < n - 2$.

Further for $\{B(x_1, r_1), \dots, B(x_q, r_q)\}$ a finite covering of K by open balls whose closure lie in $\mathcal{F}_p[N] \setminus \{x\}$ let $r(K)$ denote the minimum radius of these balls. Then for each $M \in \Gamma_p[N]$ we have that $\|G(y - M < x >)\| \leq \frac{1}{\omega_n} r(K)^{1-n}$. It now follows that the series $\sum_{M \in \Gamma_p[N] \setminus \mathcal{T}_p[N]} \|J(M, x) G(y - M < x >)\|$ is dominated by the series $\frac{1}{\omega_n} r(K)^{1-n} \sum_{M \in \Gamma_p[N] \setminus \mathcal{T}_p[N]} \|J(M, x)\|$. Consequently it follows from Lemma 2 that the subseries $\sum_{M \in \Gamma_p[N] \setminus \mathcal{T}_p[N]} J(M, x) G(y - M < x >)$ is convergent on K for $p < n - 2$.

It now follows that the series (13) is uniformly convergent on K for $p < n - 2$.

□

In fact the above proof can be readily adapted to show that the above series is uniformly convergent on any compact subset of $H^+(\mathbb{R}^n) \setminus \cup_{M \in \Gamma_p[N]} \{M < x >\}$. Moreover it may be readily seen that this series defines a right monogenic function, $C_{p,N}(x, y)$, in the variable y on the domain $H^+(\mathbb{R}^n) \setminus \cup_{M \in \Gamma_p[N]} \{M < x >\}$. Further as the function $J(M, x) G(w - M < x >)$ is left monogenic in x the function $C_{p,N}(w, x)$ is left monogenic in x .

It should be noted that it follows from the definition of a fundamental domain that the kernel $C_{p,N}(x, y)$ has precisely one singularity in the domain $\mathcal{F}_p[N]$. This singularity is of order $n - 1$.

Proposition 4 *The kernel $C_{p,N}(x, y)$ satisfies the asymmetry relation $C_{p,N}(x, y) = -\tilde{C}_{p,N}(y, x)$.*

Proof: Notice that

$$\begin{aligned}
G(y - M < x >) &= G(y - (ax + b)(cx + d)^{-1}) = G(y - \widetilde{(cx + d)}^{-1} \widetilde{(ax + b)}) \\
&= G(y - (x\tilde{c} + \tilde{d})^{-1}(x\tilde{a} + \tilde{b})) = (x\tilde{c} + \tilde{d})^{-1} \|x\tilde{c} + \tilde{d}\|^n \frac{(x\tilde{c} + \tilde{d})y - x\tilde{a} - \tilde{b}}{\|(x\tilde{c} + \tilde{d})y - x\tilde{a} - \tilde{b}\|^n} \\
&= \widetilde{(cx + d)}^{-1} \|cx + d\|^n \frac{(x\tilde{c} + \tilde{d})y - x\tilde{a} - \tilde{b}}{\|(x\tilde{c} + \tilde{d})y - x\tilde{a} - \tilde{b}\|^n}.
\end{aligned}$$

$$\text{So } J(M, x)G(y - M < x >) = \frac{(x\tilde{c} + \tilde{d})y - x\tilde{a} - \tilde{b}}{\|(x\tilde{c} + \tilde{d})y - x\tilde{a} - \tilde{b}\|^n}.$$

Now

$$\begin{aligned}
(x\tilde{c} + \tilde{d})y - x\tilde{a} - \tilde{b} &= x\tilde{c}y + \tilde{d}y - x\tilde{a} - \tilde{b} \\
&= x(\tilde{c}y - \tilde{a}) + \tilde{d}y - \tilde{b} = -x(-\tilde{c} + \tilde{a}) + \tilde{d}y - \tilde{b} \\
&= (-x + (\tilde{d}y - \tilde{b})(-\tilde{c}y + \tilde{a})^{-1})(-\tilde{c}y + \tilde{a}) = (-x + M^* < y >)(-\widetilde{yc + a})
\end{aligned}$$

where $M^* = \begin{pmatrix} \tilde{d} & -\tilde{b} \\ -\tilde{c} & \tilde{a} \end{pmatrix} \in \Gamma_p[N]$. Consequently

$$J(M, x)G(y - M < x >) = G(M^* < y > -x)\tilde{J}(M^*, y).$$

As $\star : \Gamma_p[N] \rightarrow \Gamma_p[N] : M \rightarrow M^*$ is an isomorphism it follows that $C_{p,N}(x, y) = -\tilde{C}_{p,N}(y, x)$. \square

A further property of the kernel $C_{p,N}(x, y)$ is given as follows.

Proposition 5 *The kernel, $C_{p,N}(x, y)$, satisfies the relation $J(L, x)C_{p,N}(L < x >, y) = C_{p,N}(x, y)$ for each $L \in \Gamma_p[N]$.*

Proof: Now

$$\begin{aligned}
J(M, L < x >)G(y - (ML) < x >) &= J(L, x)^{-1}J(ML, x)G(y - (ML) < x >) \\
&= J(L, x)^{-1}J(A, x)G(y - A < x >)
\end{aligned}$$

where $A = ML$. As $\Theta_L : \Gamma_p[N] \rightarrow \Gamma_p[N] : M \rightarrow ML$ is a bijection we now have that $C_{p,N}(L < x >, y) = J(L, x)^{-1}C_{p,N}(x, y)$. \square

The projection $C'_{p,N}(x', y')$ of the kernel $C_{p,N}(x, y)$ is thus an F_1 valued left monogenic section in y' that lives over the manifold $\mathcal{M}_p[N]$.

We may therefore draw the conclusion that $C_{p,N}$ is a monogenic $\Gamma_p[N]$ -periodic function. Its projection is a section on $\mathcal{M}_p[N]$ with only one point singularity. This singularity is of the order of the Cauchy kernel.

Definition 5 *A hypersurface S lying in \mathbb{R}^n is called a strongly Lipschitz surface if locally it is the graph of a real valued Lipschitz continuous function and the Lipschitz constants for these local Lipschitz graphs are bounded.*

Theorem 1 (*Cauchy's integral formula for $\mathcal{M}_p[N]$*)

Suppose $p < n - 2$. Let $U \subset F_p[N]$ be a domain and V be a bounded subdomain with a strongly Lipschitz boundary, S , lying in U . Then for each $y' \in V'$ and every left monogenic section $f' : U' \rightarrow F_1$ we have

$$f'(y') = \int_{\partial S'} \tilde{C}'_{p,N}(x', y') d\sigma'(x') f'(x'), \quad (14)$$

Proof: We can rewrite the kernel $C_{p,N}(x, y)$ as $G(x - y) + (C_{p,N}(x, y) - G(x - y))$. The term $C_{p,N}(x, y) - G(x - y)$ has no singularity on the fundamental domain $\mathcal{F}_p[N]$ while $DG(x - y) = \delta_{x=y}$, the Dirac delta function. The result follows on projecting to the manifold $\mathcal{M}_p[N]$. \square

This property has the following interpretation within the theory of automorphic forms:

Recall that all the sections that are well-defined and monogenic on the whole manifold $\mathcal{M}_p[N]$ lift to the class of automorphic forms on $\Gamma_p[N]$ that have the property of being monogenic on the whole upper half-space. In particular, it locally reproduces the projections to $\mathcal{M}_p[N]$ of the special monogenic Eisenstein series $E_{p,N}(x)$. Furthermore, it reproduces all the projections to $\mathcal{M}_p[N]$ of the general Poincaré series given in [23] of the type

$$Q(x, \tilde{f}) = \sum_{M: \Gamma_p[N] \backslash \mathcal{T}_p[N]} J(M, x) \tilde{f}(M < x >).$$

where \tilde{f} is an arbitrary bounded left monogenic function on $H^+(\mathbb{R}^n)$ that is invariant under the translation group $\mathcal{T}_p[N]$.

For $k < n$, Lemma 1 and the proof of Proposition 3 can easily be adapted to also obtain explicit formulas for kernels for k -genic sections on the manifold $\mathcal{M}_p[N]$.

More precisely we have

Proposition 6 *For $k < n$ and $p < n - 1 - k$ the series*

$$\sum_{T \in \mathcal{T}_p[N]} \sum_{M: \Gamma_p[N] \backslash \mathcal{T}_p[N]} J_k(TM, x) G_k(y - TM < x >). \quad (15)$$

is uniformly convergent on any compact subset of $\mathcal{F}_p[N] \setminus \{x\}$.

We shall denote the kernel given by expression (15) by $C_{k,p,N}(x, y)$. Note that when $k = 1$ we get back the kernel $C_{p,N}(x, y)$.

In the cases where k is even the kernel $C_{k,p,N}(x, y)$ do not vanish in the particular cases where $N = 1$ or $N = 2$. This is because each term $J_k(M, x) G_k(M < x > -y)$ arising in Series (15) equal $\frac{1}{\|cx+d\|^{n-k} \|y-M<x>\|^{n-k}}$. These are all positive terms so Series (15) cannot vanish.

By very similar arguments to those used to establish Proposition 4 it may be determined that when k is odd and $N \geq 3$ then $C_{k,p,N}(x, y) = -\tilde{C}_{k,p,N}(y, x)$ and when k is even and $N \in \mathbb{N}$ then $C_{k,p,N}(x, y) = C_{k,p,N}(y, x)$. Further by

a minor adaptation of arguments given to prove Proposition 5 it may be seen that $J_k(L, x)C'_{k,p,N}(L < x >, y) = C_{k,p,N}(x, y)$ for each $L \in \Gamma_p[N]$.

We may now consider new bundles, F_k constructed over $\mathcal{M}_p[N]$ by making the identification (x, X) with $(M < x >, J_k(M, x)X)$ for each $M \in \Gamma_p[N]$ with $x \in H^+(\mathbb{R}^n)$ and $X \in Cl_n$. Note that in the cases where k is even the conformal weight factor $J_k(M, x)$ is real valued. So in these cases the bundles are not spinor bundles.

We can now establish the following:

Theorem 2 Suppose that $\psi' : \mathcal{M}_p[N] \rightarrow F_k$ is a C^k section with compact support. Then for each $y' \in \mathcal{M}_p[N]$

$$\psi'(y') = D^k_{\mathcal{M}_p[N]} \int_{\mathcal{M}_p[N]} \tilde{C}'_{k,p,N}(y', x') \psi'(x') dm'(x') \quad (16)$$

and

$$\psi'(y') = \int_{\mathcal{M}_p[N]} \tilde{C}'_{k,p,N}(x', y') D^k_{\mathcal{M}_p[N]} \psi'(x') dm'(x') \quad (17)$$

where $C'_{k,p,N}(x', y')$ is the projection to F_k of the kernel $C_{k,p,N}(x, y)$ and m' is the projection to $\mathcal{M}_p[N]$ of Lebesgue measure on $\mathcal{F}_p[n]$. Further $p < n - 2$ and $N \geq 1$ when k is even and $N \geq 3$ when k is odd.

Proof: The section ψ' lifts to a function ψ defined on the fundamental domain $\mathcal{F}_p[N]$. This function is C^k and has compact support. Now consider $D^k \int_{\mathcal{F}_p[N]} \tilde{C}_{p,N}(x, y) \psi(x) dx^n$. This expression is equal to

$$D^k \int_{\mathcal{F}_p[N]} G_k(x - y) \psi(x) dx^n + D^k I(y) \quad (18)$$

where $I(y) = \int_{\mathcal{F}_p[N]} (\tilde{C}_{p,N}(x, y) \psi(x) - G_k(x - y) \psi(x)) dx^n$. The term $I(y)$ is left k -genic on $\mathcal{F}_p[N]$. So expression (18) reduces to $D^k \int_{\mathcal{F}_p[N]} G_k(x - y) \psi(x) dx^n$ and this term is equal to $\psi(y)$.

If instead of lifting ψ' to the fundamental domain $\mathcal{F}_p[N]$ we had lifted it to the fundamental domain $M < \mathcal{F}_p[N] >$ for some $M \in \Gamma_p[N]$ then ψ' would lift to the function $\psi(u)$ where $u \in M < \mathcal{F}_p[N] >$. Again this function is C^k with compact support in $M < \mathcal{F}_p[N] >$. In this case we would get by changing variables back to x and y

$$J_{-k}(M, y) \psi(M < y >) = D^k \int_{\mathcal{F}_p[N]} \tilde{C}_{p,N}(x, y) J_{-k}(M, x) \psi(x) dx^n,$$

where $J_{-k}(M, x) = \frac{\widetilde{cx+d}}{\|cx+d\|^{n+k+1}}$ when k is odd and $J_{-k}(M, x) = \frac{1}{\|cx+d\|^{n+k}}$ when k is even. This establishes equation (16). Similar arguments gives us equation (17). \square

It remains to set up a monogenic Cauchy kernel for the cases $p = n - 2, p = n - 1$. On placing $p = n - 1$ or $p = n - 1$ in the series for $C_{p,N}(x, y)$ this series will

now diverge. However, we can bypass this issue by again adapting the Hecke trick. For each $M \in \Gamma_p[N]$ let us write the term $\frac{x_n}{\|cx+d\|^2}$ as $H(M, x)$. It should be noted that for each $T \in \mathcal{T}_p[N]$ and each $M \in \Gamma_p[N]$ we have that $H(TM, x) = H(M, x)$. Now let us introduce the series

$$\sum_{T \in \mathcal{T}_p[N]} \sum_{M: \Gamma_p[N] \setminus \mathcal{T}_p[N]} H(TM, x)^s J(TM, x) G(y - TM < x >),$$

where s is the complex auxiliary parameter introduced in Proposition 2. As $H(TM, x) = H(M, x)$ and $J(TM, x) = J(M, x)$ one can apply the Hecke trick described in Section 3 to adapt the proof of Proposition 3 to establish the following:

Proposition 7 *The series*

$$\sum_{T \in \mathcal{T}_{n-2}[N]} \sum_{M: \Gamma_{n-2} \setminus \mathcal{T}_{n-2}[N]} H(TM, x)^s J(TM, x) G(y - TM < x >) \quad (19)$$

is absolutely convergent on any compact subset of $\mathcal{F}_{n-2}[N] \setminus \{y\}$ and for any $s \in \mathbb{C}$ whose real part is greater than zero.

Given that $H(TM, x) = H(M, x)$ and $J(TM, x) = J(M, x)$ the series (19) can be rewritten as

$$\sum_{T \in \mathcal{T}_{n-2}[N]} \sum_{M: \Gamma_{n-2} \setminus \mathcal{T}_{n-2}[N]} H(M, x)^s J(M, x) G(y - TM < x >). \quad (20)$$

Further one can combine the arguments used to establish Proposition 7 with arguments presented in [23] to obtain

Proposition 8 *The series*

$$\begin{aligned} & \sum_{M: \Gamma_{n-1}[N] \setminus \mathcal{T}_p[N]} H(M, x)^s J(M, x) G(y - M < x >) + \sum_{T_{\underline{m}}, T_{-\underline{m}} \in \mathcal{T}_{n-1}[N]} \\ & \sum_{M: \Gamma_{n-1}[N] \setminus \mathcal{T}_{n-1}[N]} H(M, x)^s J(M, x) (G(y - T_{\underline{m}} M < x >) + G(y - T_{-\underline{m}} M < x >)) \end{aligned} \quad (21)$$

is absolutely convergent on any compact subset of $\mathcal{F}_{n-1}[N] \setminus \{y\}$ and for any $s \in \mathbb{C}$ whose real part is positive.

Let us denote the series (20) and (21) by $C_{n-2, N, s}(x, y)$ and $C_{n-1, N, s}(x, y)$ respectively. From Proposition 2 it now follows that $\lim_{s \rightarrow 0} C_{n-2, N, s}(x, y)$ and $\lim_{s \rightarrow 0} C_{n-1, N, s}(x, y)$ exist and are functions $C_{n-2, N}(x, y)$ and $C_{n-1, N}(x, y)$ respectively defined on $H^+(\mathbb{R}^n) \times H^+(\mathbb{R}^n) \setminus \text{diagonal}$ where $\text{diagonal} H^+(\mathbb{R}^n) = \{(x, x) : x \in H^+(\mathbb{R}^n)\}$.

Note that for each $L \in \Gamma_p[N]$

$$H(M, L < x >) = H(ML, x).$$

Consequently from the same arguments used to prove Proposition 5 we have that $J(L, x)C_{n-2, N, s}(L < x, y) = C_{n-2, N, s}(x, y)$ and $J(L, x)C_{n-1, N, s}(L < x, y) = C_{n-1, N, s}(x, y)$ for each $L \in \Gamma_k[N]$. It follows that

$$\lim_{s \rightarrow 0} C_{p, N, s}(L < x, y) = J(L, x)^{-1} C_{p, N}(x, y)$$

for $p = n - 2$ and for $p = n - 1$. So $J(L, x)C_{p, N}(L < x, y) = C_{p, N}(x, y)$ also for $p = n - 2$ and for $p = n - 1$.

By the same argument used to prove Proposition 4 we may see that $C_{p, N, s}(y, x) = -\tilde{C}_{p, N, s}(x, y)$ for $p = n - 2$ and $p = n - 1$. Consequently $C_{p, N}(y, x) = -\tilde{C}_{p, N}(x, y)$ for $p = n - 2$ and $p = n - 1$.

Notice that for $p = n - 2$ and $p = n - 1$ the functions $C_{p, N}(x, y)$ have exactly one point singularity in each fundamental domain. This becomes clear when rewriting $C_{p, N}(x, y)$ in the equivalent form $G(x - y) + (C_{p, N}(x, y) - G(x - y))$. The term $C_{p, N}(x, y) - G(x - y)$ has no singularities on the fundamental domain $\mathcal{F}_p[N]$.

Bearing these comments in mind we can adapt arguments given in [36] and elsewhere to obtain:

Theorem 3 *Suppose that U is a bounded domain in $\mathcal{F}_p[N]$ with strongly Lipschitz boundary S lying in $\mathcal{F}_p[N]$. Suppose further that $\psi : S \rightarrow Cl_n$ belongs to $L^q(S)$ for some $q \in (1, \infty)$. Then for each smooth path $\lambda(t)$ lying in U with nontangential limit $\lambda(0) = y \in S$ we have*

$$\begin{aligned} \lim_{t \rightarrow 0} \int_{S'} \tilde{C}'_{p, N}(x', \lambda(t)') n'(x') \psi'(x') d\sigma'(x') &= \frac{1}{2} \psi'(y) \\ &+ P.V. \int_{S'} \tilde{C}'_{p, N}(x', y') n'(x') \psi'(x') d\sigma'(x') \end{aligned}$$

for almost all $y' \in S'$.

Further as the term $C_{p, N}(x, y) - G(x - y)$ is bounded on S it follows from arguments presented for instance in [36] that the singular integral

$$P.V. \int_{S'} \tilde{C}_{p, N}(x', y') n'(x') \psi'(x') d\sigma(x')$$

defines an L^q bounded operator $\Sigma_{S'} : L^q(S') \rightarrow L^q(S')$.

Similarly we have that if U , S and ψ are as in Theorem 3 but now λ is a smooth path lying in $\mathcal{F}_p[N] \setminus (U \cup S)$ with nontangential limit $\lambda(0) = y \in S$ then

$$\begin{aligned} \lim_{t \rightarrow 0} \int_{S'} \tilde{C}'_{p, N}(x', \lambda(t)') n'(x') \psi'(x') dm'(x') &= -\frac{1}{2} \psi'(y') \\ &+ PV \int_{S'} \tilde{C}'_{p, N}(x', y') n'(x') \psi'(x') dm'(x') \end{aligned}$$

for almost all $y \in S'$.

It may easily be determined by adapting arguments from [36] and elsewhere to the situation described here that the operator $\frac{1}{2}I + \Sigma_{S'}$ acting on $L^q(S')$ is a projection onto the generalized Hardy space $H^{q,+}$ of left monogenic sections on U whose nontangential maximal function on S' belongs to $L^q(S')$. Here I is the identity operator acting on $L^q(S')$. However for $p \leq n - 2$ the operator $-\frac{1}{2}I + \Sigma_{S'}$ is not necessarily a projection operator acting on $L^q(S')$.

To see this recall that in Section 3 the fundamental domain $\mathcal{F}_p[N]$ is set up so that for $p \leq n - 2$ nontrivial open subsets of $\partial H^+(\mathbb{R}^n)$ belong to $\partial \mathcal{F}_p[N]$. Suppose that w belongs to such an open subset then in general

$$\lim_{y \rightarrow w} \int_{S'} \tilde{C}'_{p,N}(x', y') n'(x') \psi'(x') d\sigma(x')$$

need not be zero.

To overcome this situation let us instead of considering the fundamental domain $\mathcal{F}_p[N]$ let us instead consider the fundamental domain of $\mathbb{R}^n \setminus \Gamma_p[N]$ which contains $\mathcal{F}_p[N]$. We shall denote this fundamental domain by $\mathcal{G}_p[N]$. It should be noted that now $w \in \mathcal{G}_p[N]$ and that if $y(t)$ is a path in $\mathcal{G}_p[N]$ that tends to infinity then $\lim_{t \rightarrow \infty} \int_S \tilde{C}_{p,N}(x, y(t)) n(x) \psi(x) d\sigma(x)$ is zero.

Let us now introduce the generalized Hardy space $H^{q,-}(S')$ of left monogenic sections defined on $\mathcal{N}_p[N] \setminus (U' \cup S')$ whose nontangential maximal function belongs to $L^q(S')$. Further $\mathcal{N}_p[N]$ is the conformally flat manifold obtained through the factorization $\mathbb{R}^n \setminus \Gamma_p[N]$. It may now be determined that the operator $-\frac{1}{2}I + \Sigma_{S'}$ is a projection operator from $L^q(S')$ onto $H^{q,-}(S')$. Consequently we have:

Theorem 4 *Suppose that S is as in Theorem 3. Then for $q \in (1, \infty)$ and $p \leq n - 2$*

$$L^q(S') = H^{q,+}(S') \oplus H^{q,-}(S').$$

One can go further than this and set up operators of Calderon-Zygmund type in this context. Suppose that $\phi : \mathbb{R}^n \rightarrow Cl_n$ is an odd, smooth function that is homogeneous of degree zero. Now consider the kernel $K(x - y) = \frac{\phi(x-y)}{\|x-y\|^n}$. For S a strongly Lipschitz surface lying in \mathbb{R}^n then provided $K(x - y)$ satisfies the usual cancellation property described in [36] and elsewhere the singular integral $P.V. \int_S K(x - y) n(x) \psi(x) d\sigma(x)$ defines an operator $K_S : L^q(S) \rightarrow L^q(S)$ of Calderon-Zygmund type. From Lemma 1 and Proposition 3 it now follows that as $K(x)$ is homogeneous of degree $1 - n$ the series

$$K_{p,N}(x, y) := \sum_{T \in \mathcal{T}_p[N]} \sum_{M \in \Gamma_p[N] \setminus \mathcal{T}_p[N]} K(TM < x > -y) J(M, x)$$

is uniformly convergent on each compact subset of $H^+(\mathbb{R}^n) \setminus \cup_{M \in \Gamma_p[N]} \{M < y >\}$. Further, by the same arguments used to establish Proposition 5 we may determine the following automorphic invariance of the kernel $K_{p,N}(x, y)$.

Proposition 9 *For each $L \in \Gamma_p[N]$ we have $J(L, x) K_{p,N}(x, y) = K_{p,N}(L < x >, y)$.*

Consequently we have:

Theorem 5 *Suppose that $p < n - 2$. Suppose also that S is a strongly Lipschitz surface lying in $\mathcal{F}_p[N]$. Then the operator $K'_{S'}$, defined by the singular integral $P.V. \int_{S'} K'(x', y') n'(x') \psi'(x') d\sigma(x')$ is L^q bounded for $q \in (1, \infty)$.*

6 k -Hypergenic functions

In this section we turn to look at analogous results in the hyperbolic setting. In the first subsection we introduce hypermonogenic Eisenstein series that project to hypermonogenic sections defined on a particular spinor bundle introduced here. However, the results produced in this first subsection automatically carry over for k -hypergenic functions and even for k -hyperbolic harmonic Eisenstein series. For this reason we treat all cases together in this subsection. In the second subsection we focus on introducing the fundamental solutions on $\mathcal{M}_p[N]$ of hypermonogenic sections and hyperbolic harmonic functions together with some of their basic properties.

6.1 k -hypergenic Eisenstein series

The simplest example of a k -hypergenic function is the constant function $F(x) = 1$. As pointed out in our preliminary section if $f(y)$ is k -hypergenic in the variable $y = M < x >$ then $K_k(M, x)f(M < x >)$ is a k -hypergenic function in the variable x . Consequently upon applying this conformal weight factor to the constant function $F(y) = 1$ and Lemma 1 one may now introduce k -hypergenic Eisenstein series in upper half space as follows:

Proposition 10 [8] *For $p < n$ and arbitrary real k with $k < n - p - 2$ the series*

$$\varepsilon_{k,p,N}(x) := \sum_{M \in \Gamma_p[N] \setminus \mathcal{T}_p[N]} K_k(M, x). \quad (22)$$

is uniformly convergent on $H^+(\mathbb{R}^n)$ and defines a k -hypergenic function.

Notice that in all cases where $k < -1$, these series converge even for $p = n - 1$.

In complete analogy to the proof of Proposition 6 one can show that

$$\varepsilon_{k,p,N}(x) = K_k(M, x) \varepsilon_{k,p,N}(M < x >) \quad \forall M \in \Gamma_p[N].$$

So the series $\varepsilon_{k,p,N}(x)$ defines a k -hypergenic Eisenstein series for the group $\Gamma_p[N]$. Furthermore, for $N \geq 3$ we have $\lim_{x_n \rightarrow +\infty} \varepsilon_{k,p,N}(e_n x_n) = 1$. This ensures, that the series $\varepsilon_{k,p,N}(x)$ are non-vanishing functions.

Following Proposition 1(i) and [8] if

$$f(x) = K_k(M, x) f(M < x >) \quad \forall M \in \Gamma_p[N],$$

is a k -hypergenic automorphic form then the function

$$g(x) := \frac{f(x)e_n}{x_n^k} = K_{-k}(M, x)g(M < x >) \quad \forall M \in \Gamma_p[N].$$

is a $-k$ -hypergenic automorphic form.

This allows us readily to construct non-vanishing k' -hypergenic Eisenstein series for positive $k' > 1$ from the k -hypergenic Eisenstein series of negative $k < -1$, simply by forming

$$E_{-k,p,N}(x) := \frac{\varepsilon_{k,p,N}(x)e_n}{x_n^k}.$$

The series $E_{-k,p,N}(x)$ then satisfy the transformation law

$$E_{-k,p,N}(x) := K_{-k}(M, x)E_{-k,p,N}(M < x >)$$

for all $M \in \Gamma_p[N]$. Since the original series $\varepsilon_{k,p,N}(x)$ are non-vanishing functions, the series $E_{-k,p,N}(x)$ do not vanish, either.

In particular, this construction provides us with non-trivial hypermonogenic Eisenstein series, which we obtain by putting $k = -n + 2$, i.e.

$$E_{n-2,p,N}(x) := \frac{\varepsilon_{-n+2,p,N}(x)e_n}{x_n^{2-n}} = \sum_{M \in \Gamma_p[N] \setminus \mathcal{T}_p[N]} \frac{1}{x_n^{2-n}} K_{-n+2}(M, x)e_n.$$

These satisfy in particular

$$E_{n-2,p,N}(x) := K_{2-n}(M, x)E_{n-2,p,N}(M < x >)$$

for all $M \in \Gamma_p[N]$.

Let us denote by E_k the particular spinor bundle over $\mathcal{M}_p[N]$ constructed by making the identification $(x, X) \leftrightarrow (M < x >, K_k(M, x)X)$ for every $M \in \Gamma_p[N]$ where $x \in H^+(\mathbb{R}^n)$ and $X \in Cl_n$. If an E_k valued section defined on a domain U' of $\mathcal{M}_k[N]$ lifts to a k -hypergenic function on the covering set U of U' then that section is called a left k -hypergenic section.

We may now state:

The projection map applied to $\varepsilon_{k,p,N}$ induces a well-defined non-vanishing k -hypergenic section with values in the spinor bundle E_k .

By similar arguments to those used to introduce k -hypergenic Eisenstein series in this section one may determine the following:

Theorem 6 *For any positive integer N , for $p < n$ and an arbitrary real k with $k < n - p - 2$ the series*

$$\mu_{k,p,N}(x) := \sum_{M \in \Gamma_p[N] \setminus \mathcal{T}_p[N]} L_k(M, x)$$

is uniformly convergent on $H^+(\mathbb{R}^n)$ and defines a k -hyperbolic harmonic function satisfying $\mu_{k,p,N}(x) = L_k(M, x)\mu_{k,p,N}(M < x >)$.

Thus for this range of p we have introduced k -hyperbolic harmonic Eisenstein series.

Following Proposition 1(i) and arguments leading to establishing Theorem 3 we now have:

Theorem 7 *For any positive integer N , for $p < n$ and an arbitrary real k with $k < n - p - 2$ the series*

$$\Theta_{k,p,N}(x) := x_n^k \sum_{M \in \Gamma_p[N] \setminus \mathcal{T}_p[N]} L_k$$

is uniformly convergent on $H^+(\mathbb{R}^n)$. Moreover, $\Theta_{k,p,N}(x)$ is a solution to the Weinstein equation $W_k u = 0$. Further $\Theta_{k,p,N}(x)$ satisfies $\Theta_{k,p,N}(x) = L_k(M, x) \Theta_{k,p,N}(M < x >)$.

It follows from [11] that $\Theta_{k,p,N}$ is a Maass wave form.

Let us denote by B_k the particular bundle over $\mathcal{M}_k[N]$ constructed by making the identification (x, X) with $(M < x >, L_k(M, x)X)$ for each $M \in \Gamma_p[N]$, with $x \in H^+(\mathbb{R}^n)$ and $X \in Cl_n$. We shall call a B_k valued section defined on an open subset U' of $\mathcal{M}_k[N]$ a k -hyperbolic harmonic section if it lifts to a k -hyperbolic harmonic function on the lifting of U .

It follows from Theorem 4 that the Eisenstein series $\Theta_{k,p,N}(x)$ projects to a well defined k -hyperbolic harmonic section defined on B_k .

6.2 hypermonogenic and hyperbolic harmonic kernels

A central aspect in the study of k -hypergenic sections on this class of manifolds is again to ask for an explicit representation of the fundamental solutions on such $\mathcal{M}_p[N]$ and for an explicit Cauchy integral formula.

The simplest case is the particular case where $k = n - 2$, the case of $(n - 2)$ -hypermonogenic functions. This is the case that we shall deal with here.

Following similar arguments to those used to establish Proposition 3 we have:

Proposition 11 *For $p = 1$*

(i) The series

$$A_{p,N}(x, y) := \sum_{T_{\underline{m}}, T_{-\underline{m}} \in \mathcal{T}_p[N] \setminus \{I\}} \sum_{M \in \Gamma_p[N] \setminus \mathcal{T}_p[N]} (K_{n-2}(T_{\underline{m}}M, x) + K_{n-2}(T_{-\underline{m}}M, x)) \\ p(M < x >, y)$$

is uniformly convergent on any compact subset of $H^+(\mathbb{R}^n) \setminus \cup_{M \in \Gamma_p[N]} \{M < y >\}$. Further the kernel $A_{p,N}(x, y)$ satisfies the asymmetry relation $A_{p,N}(x, y) = -\tilde{D}_{p,N}(y, x)$, and $K_{n-2}(L, x) A_{p,N}(L < x >, y) = A_{p,N}(x, y)$ for each $L \in \Gamma_p[N]$.

(ii) The series

$$B_{p,N}(x, y) := \sum_{T \in \mathcal{T}_p[N]} \sum_{M \in \Gamma_p[N] \setminus \mathcal{T}_p[N]} K_{2-n}(TM, x) q(TM < x >, y)$$

is uniformly convergent on any compact subset of $H^+(\mathbb{R}^n) \setminus \cup_{M \in \Gamma_p[N]} \{M < y >\}$.

In order to obtain a Cauchy integral formula for hypermonogenic functions in the context we are considering here let us first note that the kernel $p(x, y)$ is hypermonogenic in the variable x . It follows that for each $M \in \Gamma_p[N] \setminus I$ we have that

$$P\left(\int_{\partial V} p(M < x >, y) \tilde{K}_{n-2}(M, x) \frac{n(x)}{x_n^{n-2}} f(x) d\sigma(x)\right) = 0$$

for each function f which is left hypermonogenic in a neighbourhood of the closure of the bounded domain V . Further we are assuming that $M < x > \neq y$. It follows that we now have the following version of Cauchy's integral formula.

Theorem 8 *Suppose that U is a domain in upper half space satisfying $M(U) = U$ for each $M \in \Gamma_p[N]$. Suppose also that $f : U \rightarrow Cl_n$ is a left hypermonogenic function satisfying $K_{n-2}(M, x)f(M < x >) = f(x)$. Suppose further that V is a bounded subdomain of U and that the closure of V lies in a fundamental domain of $\Gamma_p[N]$. Then for each $y \in V$ and for $p < n - 2$*

$$P(f(y)) = P \int_{\partial V} \tilde{A}_{p,N}(x, y) \frac{n(x)}{x_n^{n-2}} f(x) d\sigma(x). \quad (23)$$

To obtain a complete Cauchy integral formula for all of $f(y)$ in this context let us first note from Proposition 1(i) that if $f(x)$ is right hypermonogenic then $e_n x_n^{n-2} f(x)$ is right hypermonogenic. Further in [14] it is shown that $q(x, y)$ is right $(2 - n)$ -hypergenic. Now consider the integral

$$\int_{\partial M^{-1}(V)} e_n u_n^{n-2} q(u, y) \frac{n(u)}{u_n^{n-2}} g(u) d\sigma(u)$$

where $u = M < x >$ and u_n is the n th component of u . Further g is a C^1 function defined on U , where U and V are as in Theorem 5. Under a conformal change in variables this integral becomes

$$\int_{\partial V} e_n q(M < x >, y) \tilde{K}_{2-n}(M, x) n(x) K_{n-2}(M, x) g(M < y >) d\sigma(x).$$

It follows from Proposition 1(i) that $q(M < x >, y) \tilde{K}_{2-n}(M, x)$ is right $(2 - n)$ -hypergenic in x . Consequently if U , V , y and f are as in Theorem 5 then

$$Q \int_{\partial V} q(M < x >, y) \tilde{K}_{2-n}(M, x) n(x) f(x) d\sigma(x) = 0$$

for each $M \in \Gamma_p[N] \setminus \{I\}$. Consequently we have

Theorem 9 *Suppose U , V , y and f are as in Theorem 5 and $p < n - 2$ then*

$$Q(f(y)) = Q\left(\int_{\partial V} \tilde{B}_{p,N}(x, y) n(x) f(x) d\sigma(x)\right) e_n. \quad (24)$$

Combining Theorems 5 and 6 we have

Theorem 10 *Suppose U, V, y and f are as in Theorems 5 and 6, and $p < n-2$ then*

$$f(y) = P\left(\int_{\partial V} \tilde{A}_{p,N}(x, y) \frac{n(x)}{x_n^{n-2}} f(x) d\sigma(x)\right) + Q\left(\int_{\partial V} \tilde{B}_{p,N}(x, y) n(x) f(x) d\sigma(x)\right) e_n.$$

By the same arguments used to prove Proposition 3 we may deduce:

Proposition 12 *The series*

$$E_{2-n,N}(x, y) := \sum_{T \in \mathcal{T}_p[N]} \sum_{M: \Gamma_p[N] \setminus \mathcal{T}_p[N]} K_{2-n}(TM, x) E(TM < x >, y)$$

and

$$F_{2-n,N}(x, y) := \sum_{T \in \mathcal{T}_p[N]} \sum_{M: \Gamma_p[N] \setminus \mathcal{T}_p[N]} \hat{K}_{2-n}(TM, x) F(TM < x >, y)$$

converge uniformly on each compact subset of $H^+(\mathbb{R}^n) \setminus \cup_{M \in \Gamma_p[N]} \{M < y >\}$.

By the same arguments used to prove Proposition 5 we can deduce:

Proposition 13 *The kernels $E_{2-n,N}(x, y)$ and $F_{2-n,N}(x, y)$ satisfy the relationships $K_{2-n}(L, x) E_{k,N}(L < x >, y) = E_{k,L}(x, y)$ and $\hat{K}_{2-n}(L, x) F_{2-n,N}(L < x >, y) = F_{2-n,N}(x, y)$ for each $L \in \Gamma_k[N]$.*

From results in [14] one may show that if $\psi \in L^q(S)$, with $q \in (1, \infty)$ and S a strongly Lipschitz hypersurface in $H^+(\mathbb{R}^n)$ then the integral

$$y_n^{n-2} \left(\int_S E(x, y) n(x) \psi(x) d\sigma(x) - \int_S F(x, y) \hat{n}(x) \hat{\psi}(x) d\sigma(x) \right)$$

defines a hypermonogenic function on $H^+(\mathbb{R}^n) \setminus S$. By the same arguments used in [38] it may be determined that the integral

$$\begin{aligned} & y_n^{n-2} \left(\int_S E(M < x >, y) \tilde{K}_{2-n}(M, x) n(x) \psi(x) d\sigma(x) \right. \\ & \quad \left. - \int_S F(M < x >, y) \hat{\tilde{K}}_{2-n}(M, x) \hat{n}(x) \hat{\psi}(x) d\sigma(x) \right) \end{aligned}$$

defines a hypermonogenic function on $H^+(\mathbb{R}^n) \setminus \partial V$ for each $\psi \in L^q(\partial V)$ with $q \in (1, \infty)$ and $S \subset \mathcal{F}_p[N]$. Consequently we have:

Proposition 14 *Suppose $S \subset \mathcal{F}_p[N]$ and $\psi \in L^q(\partial V)$ with $q \in (1, \infty)$. Further suppose that $p < n-2$. Then the integral*

$$y_n^{n-2} \left(\int_S \tilde{E}_{2-n,N}(x, y) n(x) \psi(x) d\sigma(x) - \int_S \tilde{F}_{2-n,N}(x, y) \hat{n}(x) \hat{\psi}(x) d\sigma(x) \right)$$

defines a hypermonogenic function on $H^+(\mathbb{R}^n) \setminus \cup_{M \in \Gamma_p[N]} \{M < y >\}$.

It is straightforward to verify using arguments given in [38] that the term

$$\begin{aligned} & K_{n-2}(M, y)(M < y >_n)^{n-2} \left(\int_{M < S >} \tilde{E}_{2-n, N}(M < x >, M < y >) n(M < x >) \right. \\ & \quad \left. \psi(M < x >) d\sigma(M < x >) \right. \\ & \quad \left. - \int_{M < S >} \tilde{F}_{2-n, N}(M < x >, M < y >) \hat{n}(M < x >) \hat{\psi}(M < x >) d\sigma(M < x >) \right) \end{aligned}$$

is equal to

$$\begin{aligned} & y_n^{n-2} \left(\int_S \tilde{E}_{2-n, N}(x, y) n(x) K_{n-2}(M, x) \psi(x) d\sigma(x) \right. \\ & \quad \left. - \int_S \tilde{F}_{2-n, N}(x, y) \hat{n}(x) \hat{K}_{n-2}(M, x) \hat{\psi}(x) d\sigma(x) \right). \end{aligned}$$

for each $M \in \Gamma_p[N]$. This establishes a conformal invariance for our Cauchy type integral.

Suppose now that w belongs to an open subset of $\partial\mathcal{F}_p[N] \cap \partial H^+(\mathbb{R}^n)$ then

$$\lim_{y \rightarrow w} y_n^{n-2} \left(\int_S \tilde{E}_{2-n, N}(x, y) n(x) \psi(x) d\sigma(x) - \int_S \tilde{F}_{p, N}(x, y) \hat{n}(x) \hat{\psi}(x) d\sigma(x) \right) = 0$$

for each $\psi \in L^q(S)$ with $q \in (1, \infty)$. Further for $y(t)$ an unbounded path in $\mathcal{F}_p[N]$ we have

$$\lim_{t \rightarrow \infty} y_n^{n-2} \left(\int_S \tilde{E}_{2-n, N}(x, y(t)) n(x) \psi(x) d\sigma(x) - \int_S \tilde{F}_{2-n, N} \hat{n}(x) \hat{\psi}(x) d\sigma(x) \right) = 0$$

for each $\psi \in L^q(S)$ with $q \in (1, \infty)$.

Let us now introduce the Hardy space $H_{n-2}^{q,+}(S')$ of left hypermonogenic sections defined on U' whose nontangential limits lie in $L^q(S')$. Further let us introduce the Hardy space $H_{n-2}^{q,-}(S')$ of left hypermonogenic sections defined on $\mathcal{M}_p[N] \setminus (U' \cup S')$ whose nontangential limits on S' lie in $L^q(S')$. Now by similar arguments to those used in [38] and in Section 4 we have

Theorem 11 For $q \in (1, \infty)$

$$L^q(S') = H_{n-2}^{q,+}(S') \oplus H_{n-2}^{q,-}(S').$$

Following arguments developed to establish Theorem 2 and results in [38] we also have:

Theorem 12 Suppose that $p < n - 2$. Further suppose $\psi' : \mathcal{M}_p[N] \rightarrow E_{2-n}$ is a C^1 function with compact support. Then for each $y' \in \mathcal{M}_p[N]$

$$\begin{aligned} \psi'(y') &= M_{n-2} y_n'^{n-2} \left(\int_{\mathcal{M}_p[N]} \tilde{E}'_{2-n}(x', y') \psi'(x') dm'(x') \right. \\ & \quad \left. - \int_{\mathcal{M}_p[N]} \tilde{F}'_{2-n}(x', y') \hat{\psi}'(x') dm'(x') \right). \end{aligned}$$

We last turn to look at a hyperbolic harmonic kernel. In [14] it is shown that the function $\frac{1}{\|x-y\|^{n-2}\|\hat{x}-y\|^{n-2}}$ is hyperbolic harmonic in the variable y . Consequently the kernel $H(x, y) := \frac{2^{n-2}}{\omega_n} \frac{x_n^{n-2}}{\|x-y\|^{n-2}\|\hat{x}-y\|^{n-2}}$ is also hyperbolic harmonic in the variable y .

Adapting arguments given in [38] it can be shown that for any C^2 function $\psi : H^+(\mathbb{R}^n) \rightarrow \mathbb{C}$ with compact support one has

$$\psi(y) = W_{2-n} \int_{H^+(\mathbb{R}^n)} H(x, y) \psi(x) dm(x)$$

for each $y \in H^+(\mathbb{R}^n)$.

Now let us introduce the series

$$H_{p,N}(x, y) := y_n^{n-2} \sum_{T \in \mathcal{T}_p[N]} \sum_{M \in \Gamma_p[N] \setminus \mathcal{T}_p[N]} L_{2-n}(TM < y >, x) H(TM < y >, x).$$

It follows from Lemma 1 and a straightforward adaptation of the proof of Proposition 3 that this series converges uniformly on any compact subset of $H^+(\mathbb{R}^n) \setminus \cup_{M \in \Gamma_p[N]} \{M < y >\}$ for $p < n - 3$ and for any N . Consequently the kernel $H_{p,N}(x, y)$ is hyperbolic harmonic in the variable y . Also as each term in the series for $H_{p,N}(x, y)$ is positive it follows that $H_{p,N}(x, y)$ does not vanish for $N = 1$ and $N = 2$. By similar arguments to those used to establish Proposition 5 it follows that $H_{p,N}(y, x) = H_{p,N}(L < y >, x)$ for each $L \in \Gamma_p[N]$.

Let us now construct a bundle over $\mathcal{M}_p[N]$ by making the identification $(x, X) \leftrightarrow (M < x >, \frac{1}{\|cx+d\|^{2n-4}} X)$. We denote this bundle by B . We now have in complete analogy to Theorem 10:

Theorem 13 *Suppose that $p < n - 3$, Further suppose $\psi' : \mathcal{M}_p[N] \rightarrow \mathbb{C}$ is a C^2 section with compact support. Then for each $y' \in \mathcal{M}_p[N]$*

$$\psi'(y') = \Delta'_{n-2} \int_{\mathcal{M}_p[N]} H'_{p,N}(x', y') \psi'(x') dm'(x')$$

where Δ'_{n-2} is the projection to $\mathcal{M}_p[N]$ of the operator Δ_{n-2} .

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